OPTIMAL DESTABILIZING VECTORS IN SOME GAUGE THEORETICAL MODULI PROBLEMS

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ABSTRACT. We show that the "classical" Harder-Narasimhan filtration of a complex vector bundle can be viewed as a limit object for the action of the gauge group, in the direction of an optimal destabilizing vector. This vector appears as an extremal value of the so called "maximal weight function". We give a complete description of these optimal destabilizing endomorphisms. Then we show that the same principle holds for another important moduli problem: holomorphic pairs (i.e. holomorphic vector bundles coupled with morphisms with fixed source). We get a generalization of the Harder-Narasimhan filtration theorem for the associated notion of τ-stability. These results suggest that the principle holds in the whole gauge theoretical framework.

Keywords. Gauge theory, maximal weight map, complex moduli problem, stability, Harder-Narasimhan filtration, moment map.

1. INTRODUCTION

Harder and Narasimhan have proved [9] that any non semistable bundle on an algebraic curve admits a unique filtration by subsheaves such that the quotients are torsion free and semistable. This is now a classical result which was generalized for reflexive sheaves on projective manifolds [18],[13], and then to any reflexive sheaf on an arbitrary compact Hermitian manifold [4],[6].

The system of semistable quotients associated to a non semistable vector bundle by the Harder-Narasimhan theorem can be interpreted as a semistable object with respect to a new moduli problem : the moduli problem for G-bundles, where G is a product $\prod_i GL(r_i)$. The motivation of this work is to find a general principle which applies to an arbitrary moduli problem: we want to associate in a *canonical way* to a non semistable object a new moduli problem and a semistable object for this new problem. One of the motivation is to find an analogous of the Harder-Narasimhan statement for other type of complex objects, for instance holomorphic bundles coupled with sections or endomorphisms.

The proper context for such a discussion is provided by the Geometric Invariant Theory (GIT) and its further developments (see [10], [14], [11], [17], [19], [7]). This approach gives indeed some powerful tools, such as the

Hilbert criterium, to deal with the stability condition. In [7], we discuss the above mentioned principle in the finite dimensional framework, for moduli problems associated with actions of a reductive group on a finite dimensional (*possibly non compact*) manifold. We look at a holomorphic action $\alpha :$ $G \times F \to F$ of a complex reductive Lie group G on a complex manifold F. We prove, that under a certain completeness assumption [19], the notion of stability *and of semistability* associated to the choice of a compact subgroup $K \subset G$ and of a K-equivariant moment map μ may be defined in term of a G-equivariant generalized maximal weight map $\lambda : H(G) \times F \to \mathbb{R}$, where $H(G) \subset \mathfrak{g}$ is the union of all subspaces of the form $i\mathfrak{k}$. In the classical GIT, this map is known as the Hilbert numerical function (see [11]). We have proved the existence and unicity (up to equivalence) of an *optimal destabilizing vector s* in the Lie algebra of G associated to any non semistable point $f \in F$. The role of this optimal vector corresponds precisely to that of so-called adapted 1-parameter subgroups in classical GIT (see [10] or [17]). We have shown that the path $t \mapsto e^{ts}f$ converges to a point f_0 which is semistable with respect to a natural action of the centralizator $Z(s)$ on submanifold of F. The assignment $f \mapsto f_0 = \lim_{t \to \infty} e^{ts}f$ is the generalization of the Harder-Narasimhan statement.

Therefore the general principle may be stated as follows: to get an analogue of the Harder-Narasimhan result for a given complex moduli problem, one has to give a gauge theoretical formulation of the problem (i.e. describe it in term of an action of a gauge group on certain complex variety), to compute the maximal weight map and to study the optimal destabilizing vectors of the non semistable objects. Then the new semistable object is obtained as a limit in the direction of the optimal vector.

In gauge theory, there are well-known links between the stability theory of vector bundles and the GIT concepts. The analogue of the maximal weight map in this setting has already been exploited by several authors, in particular to study the Kobayashi-Hitchin correspondance (see for example [15], [12]). In the case of plain bundles, the relation between the algebraic geometric notion of stability and the Morse theory associated to the Yang-Mills functional ([1],[8]) is also well-known. *But what appears to be unavailable in the literature, even in the case of plain bundles, is the very natural fact that the extremal value of the maximal weight map in the gauge theoretical framework corresponds exactly to the optimal direction which leads to the Harder-Narasimhan filtration (or precisely to the associated system of semistable quotients).*

The purpose of the present paper is to give a direct and complete proof of this fact in two classical gauge theoretical problems : holomorphic vector bundles and holomorphic pairs (i.e. vector bundles coupled with morphisms with fixed source). These results were announced without proof in [12].

We first focus on the moduli problem of complex vector bundles: we prove that one can associate to any non semistable bundle a maximal destabilizing element in the formal Lie algebra of the gauge group, then we give a complete description of this optimal vector. The main tools here are the notion of

Harder-Narasimhan polygonal line (see [5]) and a well defined notion of energy for such lines. The system of quotients of the "classical" Harder-Narasimhan filtration appears as the limit in the direction of this optimal vector .

In a second part, we give an analogue description for the moduli problem associated to holomorphic pairs. Here the corresponding notion of stability, called τ-*stability* (see [2]), depends on the choice of a real parameter. We prove, in this setting, a generalization of the Harder-Narasimhan filtration theorem for the τ-stability (let us remark that a generalization of the Seshadri filtration in the context of τ-stability was given a few years ago in [3]). Using an explicit formula for the maximal weight function, we describe the optimal destabilizing vectors. Once again, the filtration is obtained as a limit in the direction of this extremal value of the maximal weight map.

These two results suggest that the principle holds in the infinite dimensional gauge theoretical framework. It gives an intuitive way to build the "Harder-Narasimhan object" for a given moduli problem.

2. Holomorphic vector bundles

We are interested here in the following gauge theoretical moduli problem: classifying the holomorphic structures on a given complex vector bundle up to gauge equivalence. For clarity purposes, we will assume that the base manifold is a complex curve to avoid complications related to singular sheaves (and so the use of L_1^2-Sobolev sections). Besides, the completeness property used in [7] is *formally* satisfied for *linear* moduli problems on a complex curve, so this is natural to work first in this framework.

Let E be a complex vector bundle of rank r over the Hermitian curve (Y, g). We denote by $\mathcal{G}$ the complex gauge group $\mathcal{G} := \mathrm{Aut}(E)$ whose formal Lie algebra is $A^0(\mathrm{End}(E))$.

Let h be any Hermitian structure on E and let us denote by

$$\mathcal{K}_h := U(E, h) \subset \mathcal{G}$$

the real gauge group of unitary automorphisms of E with respect to h.

We will use here the terminology developed in [19] and [7]. An element $s \in A^0(\mathrm{End}(E))$ is said to be of *Hermitian type* if there exists a Hermitian metric h on E, such that $s \in A^0(\mathrm{Herm}(E, h))$.

We identify a holomorphic structure $\mathcal{E}$ on E with the corresponding integrable semiconnections $\overline{\partial}_{\mathcal{E}}$ on E (see [12]). We are concerned with the stability theory for the action of $\mathcal{G}$ on the space $\mathcal{H}(E)$ of holomorphic structures. Fixing a Hermitian metric h, the moment map for the induced $\mathcal{K}_h$-action on $\mathcal{H}(E)$ is given by :

$$\mu(\mathcal{E}) = \Lambda_g(F_{\mathcal{E}, h}) + \frac{2\pi i}{\mathrm{vol}_g(Y)} m(\mathcal{E}) \mathrm{id}_E$$

where $F_{\mathcal{E}, h}$ is the curvature of the Chern connection associated to $\overline{\partial}_{\mathcal{E}}$ and h and

$$m(\mathcal{E}) := \frac{\deg(\mathcal{E})}{r}$$

is the slope of $\mathcal{E}$.

Let us recall that a holomorphic vector bundle $\mathcal{E} \in \mathcal{H}(E)$ is semistable with respect to this moment map if and only if it is semistable in the sense of Mumford ([14], [12]):

$m(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\operatorname{rang}(\mathcal{F})} \leqslant \frac{\deg(\mathcal{E})}{\operatorname{rang}(\mathcal{E})} := m(\mathcal{E})$ *for all reflexive subsheaves* $\mathcal{F} \subset \mathcal{E}$, *such that* $0 < \operatorname{rang}(\mathcal{F}) < r$.

One may give an analytic Hilbert type criterion for the stability theory associated to the moment map μ. We will need the following notation : if f is an endomorphism of a vector space V, we will put for any $a \in \mathbb{R}$

$$V_f(a) := \bigoplus_{a' \leqslant a} \operatorname{Eig}(f, a').$$

We extend the notation for endomorphisms of E with constant eigenvalues in an obvious way.

Then, one has an explicit formula for the maximal weight map λ ([16]) : if $\mathcal{E} \in \mathcal{H}(E)$ and $s \in \operatorname{Herm}(E, h)$ then

$$\lambda^s(\mathcal{E}) = \begin{cases} \lambda_k \deg(\mathcal{E}) + \sum_{i=1}^{k-1}(\lambda_i - \lambda_{i+1})\deg(\mathcal{E}_i) - \frac{\deg(\mathcal{E})}{r}\operatorname{Tr}(s) \\ \text{if the eigenvalues } \lambda_1 < \cdots < \lambda_k \text{ of } s \text{ are constant and} \\ \mathcal{E}_i := \mathcal{E}_s(\lambda_i) \text{ are holomorphic} \\ \\ \infty \text{ if not .} \end{cases}$$

Stability criterion: *A point* $\mathcal{E} \in \mathcal{H}(E)$ *is semistable if and only if* $\lambda^s(\mathcal{E}) \geqslant 0$ *for all* $s \in A^0(\operatorname{Herm}(E, h))$.

Let us come to the definition of optimal destabilizing endomorphisms :

Proposition 2.1. *Let* $\mathcal{E}$ *be a non semistable bundle. There exists a Hermitian endomorphism* $s_{op} \in A^0(\operatorname{Herm}(E, h))$ *such that*

$$\lambda^{s_{op}}(\mathcal{E}) = \inf_{\substack{s \in A^0(\operatorname{Herm}(E,h)) \\ \|s\|=1}} \lambda^s(\mathcal{E})$$

Proof. It is sufficient to consider the $s \in A^0(\operatorname{Herm}(E, h))$ with constant eigenvalues $\lambda_1 < \cdots < \lambda_k$ and such that the $\mathcal{E}_i$ are holomorphic. The condition $\|s\| = 1$ implies that the eigenvalues λ_i are bounded. Moreover, we know that the degree of a subbundle of $\mathcal{E}$ is bounded above ([4] prop 2.2), so that, writing λ in the form

$$\lambda^s(\mathcal{E}) = \deg(\mathcal{E})\Big[\lambda_k - \frac{1}{r}\sum_{i=1}^{k} r_i\lambda_i\Big] + \sum_{i=1}^{k-1} \deg(\mathcal{E}_i)(\lambda_i - \lambda_{i+1}),$$

it becomes obvious that $\lambda^s(\mathcal{E})$ is bounded from below on the sphere $\|s\| = 1$.

Now let $(s_n)_n$ be a sequence of Hermitian endomorphisms such that

$$\lim_{n \to +\infty} \lambda^{s_n}(\mathcal{E}) = \inf_{\substack{s \in A^0(\operatorname{Herm}(E,h)) \\ \|s\|=1}} \lambda^s(\mathcal{E}).$$

We always assume that the associated filtrations are holomorphic. Going to a subsequence if necessary, we may suppose that each s_n admits k distinct eigenvalues $\lambda_1^n < \cdots < \lambda_k^n$.

Let us recall the following result which is a direct consequence of the convergence theorem for subsheaves proved in [6] (see also [4] prop 2.9):

Proposition 2.2. *Let $(\mathcal{F}_n)_n$ be a sequence of subsheaves of $\mathcal{E}$ and assume that there exists a constant $c \in \mathbb{R}$ such that for all n*

$$\deg(\mathcal{F}_n) \geqslant c$$

Then we may extract a subsequence $(\mathcal{F}_m)_m$ which converges in the sense of weakly holomorphic subbundles to a subsheaf $\mathcal{F}$ of E. In particular

$$\limsup_{m\to+\infty} \deg(\mathcal{F}_m) \leqslant \deg(\mathcal{F}).$$

Using this proposition and the fact that the λ_i^n are bounded, one can easily extract from $(s_n)_n$ a subsequence $(s_m)_m$ such that :

  (i) *there exist indices $0 = j_0 < j_1 < \cdots < j_l = k$ and distinct values $\lambda_{j_1} < \cdots < \lambda_{j_l}$ such that for all $i \in \{j_p, \cdots, j_{p+1}\}$, $\lambda_i^m \to \lambda_{j_{p+1}}$;*
  (ii) *there exists a filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_{j_1} \subset \cdots \subset \mathcal{E}_{j_l} = \mathcal{E}$ such that each $\mathcal{E}_{j_p}$ is the limit in the sense of weakly holomorphic subbundles of the $\mathcal{E}_{j_p}^m$, which implies that $\deg(\mathcal{E}_{j_p}) \geqslant \limsup_{m\to+\infty} \deg(\mathcal{E}_{j_p}^m)$.*

For the second point, we use the fact that inclusion of subsheaves is preserved when going through the limit in the sense of weakly holomorphic subbundles [4].

Let s be the Hermitian endomorphism whose eigenvalues are the λ_{j_p} with corresponding filtration $\{\mathcal{E}_{j_p}\}$ (s may have less than k distinct eigenvalues). Then we have

$$\lambda^s(\mathcal{E}) \leqslant \liminf_{m\to+\infty} \lambda^{s_m}(\mathcal{E}) = \inf_{\substack{s\in A^0(\mathrm{Herm}(E,h)) \\ \|s\|=1}} \lambda^s(\mathcal{E})$$

so that s is an optimal destabilizing endomorphism of $\mathcal{E}$. ∎

Let us come to the proof of the result stated in [7] :

Theorem 2.3. *Let $\mathcal{E} \in \mathcal{H}(E)$ be a non semistable bundle. Then it admits a unique optimal destabilizing element $s_{op} \in A^0(\mathrm{Herm}(E,h))$ which is given by*

$$s_{op} = \frac{1}{\sqrt{\mathrm{Vol}(Y)}\sqrt{\sum_{i=1}^{k} r_i \left[\frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i} - \frac{\deg(\mathcal{E})}{r}\right]^2}} \sum_{i=1}^{k} \left[\frac{\deg(\mathcal{E})}{r} - \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i}\right] \mathrm{id}_{F_i} \,,$$

where

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

is the Harder-Narasimhan filtration of $\mathcal{E}$, F_i is the h-orthogonal complement of $\mathcal{E}_{i-1}$ in $\mathcal{E}_i$ and $r_i := \mathrm{rang}(\mathcal{E}_i/\mathcal{E}_{i-1})$.

Proof. Let $s \in A^0(\mathrm{Herm}(E,h))$ such that s has constant eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ and the filtration given by the $\mathcal{E}_i = \mathcal{E}_s(\lambda_i)$ is holomorphic. Let us denote by

$$m_i := \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i} \, , 1 \leqslant i \leqslant k, \quad m := m(\mathcal{E})$$

the slopes of the associated quotient sequence. The expression of the maximal weight map becomes :

$$\lambda^s(\mathcal{E}) = \sum_{i=1}^k \lambda_i r_i (m_i - m).$$

We want to minimize this expression with respect to s under the assumption

$$\|s\| = \sqrt{\int_Y \mathrm{Tr}(ss^\star).vol_g} = \sqrt{\mathrm{Vol}(Y)} \sqrt{\sum_{i=1}^k r_i \lambda_i^2} = 1.$$

Assume first that the filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$ is fixed and hence that the r_i and the m_i are fixed. We have to minimize the map

$$g_{\{\mathcal{E}_i\}} : (\lambda_1, \cdots, \lambda_k) \mapsto \sum_{i=1}^k \lambda_i r_i (m_i - m)$$

over the smooth ellipsoid

$$S = \big\{ (\lambda_1, \cdots, \lambda_k) \mid \sum_{i=1}^k r_i \lambda_i^2 = 1/\mathrm{Vol}(Y) \big\}.$$

with the additional open condition

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k. \quad (\star)$$

Resolving the Lagrange problem for $g|_S$ we see that there are two critical points of g over S which are obtained for

$$\lambda_i = \varepsilon(m_i - m), \ 1 \leqslant i \leqslant k \ \text{and} \ \varepsilon = \pm 1 \frac{1}{\sqrt{\mathrm{Vol}(Y)}\sqrt{\sum_{i=1}^k r_i (m_i - m)^2}}.$$

Let us remark that $g_{\{\mathcal{E}_i\}}$ is negative only for $\varepsilon < 0$.

We need the following

Lemma 2.4. *The filtration $\{\mathcal{E}_i\}_{1 \leqslant i \leqslant k}$ associated to any optimal destabilizing element s_{op} satisfies $m_1 > m_2 > \cdots > m_k$.*

Proof. If the sequence $(m_i)_{1 \leqslant i \leqslant k}$ is not decreasing, the critical point of $g_{\{\mathcal{E}_i\}}$ which may correspond to a minimum does not satisfy the condition $(\star)$. So let s_{op} be an optimal destabilizing element and $\lambda_1 < \cdots < \lambda_k$ its eigenvalues. Assume that for the corresponding filtration the sequence $(m_i)_{1 \leqslant i \leqslant k}$ is not decreasing, then $\nu = (\lambda_1, \cdots, \lambda_k)$ is not a critical point of $g = g_{\{\mathcal{E}_i\}}$, so that the gradient $\mathrm{grad}_\nu(g|_S)$ is non zero. Therefore, moving slightly the point ν in the opposite direction, we may get a new point $\nu' \in S$ which still satisfies the open condition $(\star)$ and with $g(\nu') < g(\nu)$. Thus, the corresponding Hermitian endomorphism s' satisfies $\lambda^{s'}(\mathcal{E}) < \lambda^{s_{op}}(\mathcal{E})$ which is a contradiction. ∎

Keeping in mind this result, it is sufficient to consider endomorphisms whose associated filtration $\{\mathcal{E}_i\}_{1\leqslant i\leqslant k}$ satisfies $m_1 > m_2 > \cdots > m_k$. We will call such a filtration an *admissible filtration*.

For admissible filtrations, the map $g_{\{\mathcal{E}_i\}}$ reaches its minimum for

$$(\lambda_1, \cdots, \lambda_k) = \frac{1}{\sqrt{\mathrm{Vol}(Y)}\sqrt{\sum_{i=1}^{k} r_i(m_i - m)^2}}(m - m_1, \cdots, m - m_k)$$

and

$$g_{\{\mathcal{E}_i\}}(\lambda_1, \cdots, \lambda_k) = -\frac{1}{\sqrt{\mathrm{Vol}(Y)}}\sqrt{\sum_{i=1}^{k} r_i(m_i - m)^2}.$$

Thus, we have to maximize $\sum_{i=1}^{k} r_i(m_i - m)^2$ among all admissible filtrations.

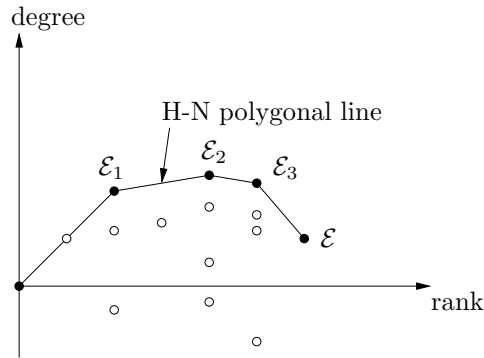Let us remind some fundamental property of the Harder-Narasimhan filtration (see [5], [4] for details):

Proposition 2.5. *The Harder-Narasimhan filtration is the unique filtration* $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$ *such that*

    (i) *each quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable for $1 \leqslant i \leqslant l$;*

    (ii) *the slope sequence satisfies*

$$m(\mathcal{E}_i/\mathcal{E}_{i-1}) < m(\mathcal{E}_{i+1}/\mathcal{E}_i) \quad \text{for } 1 \leqslant i \leqslant l - 1.$$

We may associate to any filtration $\mathcal{D} = (0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \mathcal{E}_l = \mathcal{E})$ a polygonal line $\mathcal{P}(\mathcal{D})$ in $\mathbb{R}^2$ defined by the sequence of points $p_i = (\mathrm{rang}\mathcal{E}_i, \deg\mathcal{E}_i)$ for $0 \leqslant i \leqslant l$. The line associated to the Harder-Narasimhan filtration is called *the Harder-Narasimhan polygonal line*.

Proposition 2.6 ([5])**.** *For any subsheaf $\mathcal{F}$ of $\mathcal{E}$, the point $(\mathrm{rang}\mathcal{F}, \deg\mathcal{F})$ is located below the Harder-Narasimhan polygonal line. As a consequence, any polygonal line associated to a filtration of $\mathcal{E}$ is located below the Harder-Narasimhan polygonal line.*

For any filtration $\mathcal{D} = \{\mathcal{E}_i\}_{1\leqslant i\leqslant k}$, the condition $m_1 > \cdots > m_k$ is equivalent to the fact that the line $\mathcal{P}(\mathcal{D})$ is concave and let us remark that it is satisfied by the Harder-Narasimhan filtration. Now, the expression $\sum_{i=1}^{k} r_i(m_i - m)^2$ can be interpreted as an energy of the corresponding

polygonal line. Indeed, let us denote by $f_\mathcal{D} \in C^0([0, r], \mathbb{R})$ the map whose graph is the polygonal line $\mathcal{P}(\mathcal{D})$; this is a piecewise affine function and so let $a_0 = 0 < a_1 < \cdots < a_l = r$ be the corresponding partition of $[0, r]$. We define

$$E(f_\mathcal{D}) := \sum_{i=1}^{k} r_i(m_i - m)^2 = \sum_{i=0}^{l-1} \int_{a_i}^{a_{i+1}} (f_\mathcal{D}'(x) - m)^2 \mathrm{d}x$$

Proposition 2.7. *Let $\mathfrak{D}$ be the Harder-Narasimhan filtration of $\mathcal{E}$ and $\mathcal{D}$ be any other admissible filtration, then $E(f_\mathfrak{D}) > E(f_\mathcal{D})$. The maximum of the energy among all concave polygonal lines associated to a filtration of $\mathcal{E}$ is obtained for the Harder-Narasimhan polygonal line.*

Proof. Let $\mathfrak{D}$ the Harder-Narasimhan filtration of $\mathcal{E}$ and $\mathcal{D}$ any other filtration whose polygonal line is concave. This is essentially a result about concave piecewise affine maps :

Lemma 2.8. *Let $f, g : [0, r] \to \mathbb{R}$ be two continuous piecewise affine concave functions. Assume that $g(0) = f(0)$ and $f(r) = g(r)$. If for all $x \in [0, r]$, $f(x) \geqslant g(x)$ then $E(f) \geqslant E(g)$. Equality may occur if and only if $f = g$.*
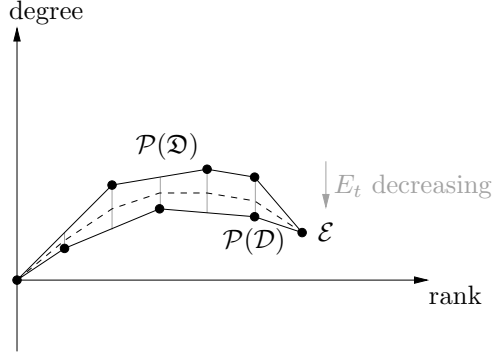
Proof. Let $a_0 = 0 < a_1 < \cdots < a_l = r$ be a partition of $[0, r]$ common to f and g. Let $F(t, x) = tf(x) + (1 - t)g(x)$ then $x \to F(t, x)$ is continuous, concave, and affine on each segment $[a_i, a_{i+1}]$. So we may define for each t the energy $E(F(t, x))$ of the map $x \mapsto F(t, x)$.

One has

$$\frac{d}{dt} E(F_t) = \sum_{i=0}^{l-1} 2 \int_{a_i}^{a_{i+1}} \frac{d^2}{dtdx} F(t, x) (\frac{d}{dx} F(t, x) - m) \mathrm{d}x$$

$$= \sum_{i=0}^{l-1} [\frac{d}{dt} F(t, x) (\frac{d}{dx} F(t, x) - m)]_{a_i}^{a_{i+1}}$$

$$- 2 \sum_{i=0}^{l-1} \int_{a_i}^{a_{i+1}} \frac{d}{dt} F(t, x) \frac{d^2}{dx^2} F(t, x) \mathrm{d}x$$

$$\geqslant \sum_{i=1}^{l-1} \big[(f(a_i) - g(a_i)) \times$$

$$(t((f)_g'(a_i) - (f)_d'(a_i)) + (1 - t)((g)_g'(a_i) - (g)_d'(a_i))) \big]$$

$$> 0$$

Indeed, the concavity of each polygonal line implies that its derivative is decreasing, that $\frac{d^2}{dx^2} F(t, x) \leqslant 0$ and we have of course $(f(a_i) - g(a_i)) \geqslant 0$. Let us remark that if $f \neq g$ the strict inequality $E(f) > E(g)$ obviously occurs. ∎

Then, by proposition 2.6 $\mathcal{P}(\mathcal{D})$ lies below $\mathcal{P}(\mathfrak{D})$, that is for any $x \in [0,r]$, $f_{\mathcal{D}}(x) \leqslant f_{\mathfrak{D}}(x)$ with strict inequality on an open subset. We use the lemma to conclude.

Summarising the results, we have proved that $\lambda^s(\mathcal{E})$ reaches its minimum for a Hermitian element $s \in A^0(\mathrm{Herm}(E,h))$ whose associated filtration is the Harder-Narasimhan filtration of $\mathcal{E}$ and whose eigenvalues are $\lambda_i = \dfrac{m-m_i}{\sqrt{\mathrm{Vol}(Y)}\sqrt{\sum_{i=1}^k r_i(m_i-m)^2}}$. Moreover it follows from the proof that this is a strict minimum.

Now using the identification of $\mathcal{H}(E)$ with the space of integrable semi-connections, it is not difficult (see for instance [15] lemma 2.3.2) to show that

$$\lim_{t \to +\infty} (e^{ts_{op}}).\overline{\partial}_{\mathcal{E}} = \overline{\partial}_{F_1} \oplus \cdots \oplus \overline{\partial}_{F_k}.$$

In other words the holomorphic structure $e^{ts_{op}}\mathcal{E}$ converges to the direct sum holomorphic structure $\bigoplus_{i=1}^k \mathcal{E}_i/\mathcal{E}_{i-1}$ as $t \to +\infty$.

This illustrates our principle that the Harder-Narasimhan filtration is a limit object precisely in the direction given by the extremal value of the maximal weight map. The direct sum holomorphic structure is of course semistable with respect to the smaller gauge group $\prod_{i=1}^k \mathrm{Aut}(E_i/E_{i-1})$.

3. Holomorphic pairs

Here we will give an analogous result for a different moduli problem associated to holomorphic pairs.

Let $\mathcal{F}_0$ be a fixed holomorphic vector bundle of rank r_0 with a fixed Hermitian metric h_0 and E a complex vector bundle of rank r on the Hermitian curve (Y,g). We are interested in the following moduli problem: classifying the holomorphic pairs $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a holomorphic structure on E and φ is a holomorphic morphism $\varphi : \mathcal{F}_0 \to \mathcal{E}$. Such a pair will be called a holomorphic pair of type $(E, \mathcal{F}_0)$ and we will denote by $\mathcal{H}(E, \mathcal{F}_0)$ the space of such pairs. Here the complex gauge group is once again the group $\mathcal{G} := \mathrm{Aut}(E)$.

Let us fix a Hermitian metric h on E, and let us denote by $\mathcal{K}_h := U(E,h)$ the group of unitary automorphisms. The moment map for the $\mathcal{K}_h$ action on $\mathcal{H}(E, \mathcal{F}_0)$ has the form :

$$\mu(\mathcal{E}, \varphi) = \Lambda_g F_{\mathcal{E},h} - \frac{i}{2}\varphi \circ \varphi^* + \frac{i}{2}t\mathrm{id}_E .$$

In the sequel we will assume that the topological condition $\mu(\mathcal{E}) \geqslant \tau$ holds.

Then we have the following characterisation of semistable pairs $(\mathcal{E}, \varphi)$ (see [2]): let $\tau := \frac{1}{4\pi} t \mathrm{Vol}_g(Y)$, then $(\mathcal{E}, \varphi)$ is semistable with respect to the moment map μ if and only if it is τ-*semistable* in the following sense :

(i) $m(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \leqslant \tau$ for all reflexive subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < r$.

(ii) $m(\mathcal{E}/\mathcal{F}) := \frac{\deg(\mathcal{E}/\mathcal{F})}{\mathrm{rk}(\mathcal{E}/\mathcal{F})} \geqslant \tau$ for all reflexive subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < r$ and $\varphi \in H^0(\mathrm{Hom}(\mathcal{F}_0, \mathcal{F}))$.

Now we may give an analogue of the Harder-Narasimhan theorem for this notion of stability:

Theorem 3.1. *Let $(\mathcal{E}, \varphi)$ be a non τ-semistable holomorphic pair of type $(E, \mathcal{F}_0)$. Then there exists a unique holomorphic filtration with torsion free quotients*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m \subset \mathcal{E}_{m+1} \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

of $\mathcal{E}$ such that:

(i) *The slopes sequence satisfies:*

$$m(\mathcal{E}_1/\mathcal{E}_0) > \cdots > m(\mathcal{E}_m/\mathcal{E}_{m-1}) > \tau > m(\mathcal{E}_{m+2}/\mathcal{E}_{m+1}) > \cdots > m(\mathcal{E}_k/\mathcal{E}_{k-1}) \ .$$

and the additional condition

$$\tau \geq m(\mathcal{E}_{m+1}/\mathcal{E}_m)$$

(ii) *The quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ are semistable for $i \neq m$.*

(iii) *One of the following properties holds:*

(a)

$$\mathrm{Im}(\varphi) \not\subset \mathcal{E}_m \ , \ \tau > \frac{\deg(\mathcal{E}_{m+1}/\mathcal{E}_m)}{\mathrm{rk}(\mathcal{E}_{m+1}/\mathcal{E}_m)}$$

and the pair $(\mathcal{E}_{m+1}/\mathcal{E}_m, \bar{\varphi})$ is τ-semistable, where $\bar{\varphi}$ is the $\mathcal{E}_{m+1}/\mathcal{E}_m$-valued morphism induced by φ.

(b)

$$\mathrm{Im}(\varphi) \not\subset \mathcal{E}_m \ , \ \tau = \frac{\deg(\mathcal{E}_{m+1}/\mathcal{E}_m)}{\mathrm{rk}(\mathcal{E}_{m+1}/\mathcal{E}_m)}$$

and $\mathcal{E}_{m+1}/\mathcal{E}_m$ is semistable of slope τ. This implies that the pair $(\mathcal{E}_{m+1}/\mathcal{E}_m, \bar{\varphi})$ is τ-semistable.

(c) $\mathrm{Im}(\varphi) \subset \mathcal{E}_m$ *and $\mathcal{E}_{m+1}/\mathcal{E}_m$ is semistable.*

Moreover, in the cases (b) and (c) the obtained filtration coincides with the "classical" Harder-Narasimhan filtration of $\mathcal{E}$ and the additional condition $m(\mathcal{E}_{m+1}/\mathcal{E}_m) > m(\mathcal{E}_{m+2}/\mathcal{E}_{m+1})$ holds.

Proof. We will use the results and methods of theorem 3.2 in [4]. In order to build the filtration, let us first consider type (i) destabilizing subsheaves of $\mathcal{E}$, i.e. subsheaves $\mathcal{F}$ which satisfy $m(\mathcal{F}) > \tau$. If there exists such a subsheaf, we let $\mathcal{E}_1$ be the maximal destabilizing type (i) subsheaf: it is nothing but the first element of the Harder-Narasimhan filtration of $\mathcal{E}$. If $\mathrm{Im}(\varphi) \subset \mathcal{E}_1$ then we follow with the classical Harder-Narasimhan filtration (case (c)), if not we consider the pair $(\mathcal{E}/\mathcal{E}_1, \varphi_1)$ where φ_1 is induced by φ and we follow the same principle until there is no more type (i) destabilizing subsheaves: we get a sequence

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m$$

which coincides with the first terms of the usual Harder-Narasimhan filtration and such that $\mathcal{E}/\mathcal{E}_m$ has no type (i) destabilizing subsheaf.

Then we consider type (ii) destabilizing subsheaves and we take the subsheaf $\mathcal{F}_1$ containing $\mathrm{Im}(\varphi)$ and minimizing the slope $\mu(\mathcal{E}/\mathcal{F}_1)$ (we take $\mathcal{F}_1$ of maximal rank with this property, it exists and it is unique by the arguments developed in [4]). This will be the *last* term of our filtration. By definition, we have $\mu(\mathcal{E}/\mathcal{F}_1) < \tau$. Moreover $\mathcal{F}_1$ contains $\mathcal{E}_m$ by the following lemma.

Lemma 3.2. *Let $\mathcal{G} \subset \mathcal{E}$ be a maximal destabilizing subsheaf of type (i) and let $\mathcal{F} \subset \mathcal{E}$ be a type (ii) maximal destabilizing subsheaf, then $\mathcal{G} \subset \mathcal{F}$.*

Proof. Assume first that $\mathcal{G} \cap \mathcal{F} = 0$, then $(\mathcal{F} + \mathcal{G})/F \simeq \mathcal{G}$ and of course $\mathrm{Im}(\varphi) \subset \mathcal{G} + \mathcal{F}$. Then we have

$$\mu((\mathcal{F} + \mathcal{G})/\mathcal{F}) = \mu(\mathcal{G}) > \tau > \mu(\mathcal{E}/\mathcal{F})$$

and using the following exact sequence

$$0 \to (\mathcal{F} + \mathcal{G})/\mathcal{F} \to \mathcal{E}/\mathcal{F} \to \mathcal{E}/(\mathcal{F} + \mathcal{G}) \to 0$$

we get $\mu(\mathcal{E}/(\mathcal{F} + \mathcal{G})) < \mu(\mathcal{E}/\mathcal{F})$ which is a contradiction. Now assume that $\mathcal{F} \cap \mathcal{G}$ is a non trivial subsheaf of $\mathcal{G}$ then using the following exact sequence

$$0 \to \mathcal{G} \cap \mathcal{F} \to \mathcal{G} \to \mathcal{G}/\mathcal{G} \cap \mathcal{F} \to 0$$

and the isomorphism $(\mathcal{G} + \mathcal{F})/\mathcal{F} \simeq \mathcal{G}/\mathcal{G} \cap \mathcal{F}$, we get

$$\mu((\mathcal{F} + \mathcal{G})/\mathcal{F}) = \mu(\mathcal{G}/\mathcal{G} \cap \mathcal{F}) < \mu(\mathcal{E}/\mathcal{F}) < \tau < \mu(\mathcal{G})$$

and $\mu(\mathcal{G} \cap \mathcal{F}) > \mu(\mathcal{G})$ which is a contradiction. So we get $\mathcal{G} \cap \mathcal{F} = \mathcal{G}$ so that $\mathcal{G} \subset \mathcal{F}$. ∎

Remark 3.3. This lemma simply states that one can always build a Harder-Narasimhan filtration starting either from its first term or from its last term.

Following the process, we get a sequence $\mathcal{F}_l \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{E}$ where $\mathcal{F}_l$ does contain $\mathcal{E}_m$ and admits no type (ii) destabilizing subsheaf. It is quite easy to prove that each quotient $\mathcal{F}_i/\mathcal{F}_{i+1}$ is semistable. Moreover, it is clear that $\mathcal{F}_l/\mathcal{E}_m$ has no type (i) destabilizing subsheaf. So, putting things together, we get a filtration

$$0 \subset \mathcal{E}_0 \subset \cdots \subset \cdots \mathcal{E}_m \subset \mathcal{E}_{m+1}(= \mathcal{F}_l) \subset \cdots \subset \mathcal{E}_{k-1} = (\mathcal{F}_1) \subset \mathcal{E}_k = \mathcal{E}$$

where $\mathrm{Im}(\varphi) \subset \mathcal{E}_{m+1}$.

Also we have

$$m(\mathcal{E}_1/\mathcal{E}_0) > \cdots > m(\mathcal{E}_m/\mathcal{E}_{m-1}) > \tau$$

and

$$m(\mathcal{E}_{m+2}/\mathcal{E}_{m+1}) > \cdots > m(\mathcal{E}_k/\mathcal{E}_{k-1}) \ .$$

If $\mathrm{Im}(\varphi) \subset \mathcal{E}_m$, then clearly the filtration coincides with the classical Harder-Narasimhan filtration and we are in case (c) of the theorem. Else the pair $(\mathcal{E}_{m+1}/\mathcal{E}_m, \overline{\varphi})$ is by construction τ-semistable with $m(\mathcal{E}_{m+1}/\mathcal{E}_m) \leqslant \tau$. Let us remark to conclude that if $m(\mathcal{E}_{m+1}/\mathcal{E}_m) = \tau$, then the notion of τ-semistability and semistability coincide for $\mathcal{E}_{m+1}/\mathcal{E}_m$ (case (b) of the theorem).

Unicity is proved in the same way as in the algebraic "classical" case (see [4] or [5]). ∎

Remark 3.4. Theorem 3.1 clearly holds for any bundle over a compact Hermitian manifold (X, g) where g is a Gauduchon metric (see [4]). Indeed, we did not use the fact that Y is one dimensional in this proof.

Coming back to our gauge moduli problem, one can once again give a formula for the maximal weight function:

$$
\lambda^s(\mathcal{E}) = \begin{cases} \lambda_k \mathrm{deg}(\mathcal{E}) + \sum_{i=1}^{k-1}(\lambda_i - \lambda_{i+1})\mathrm{deg}(\mathcal{E}_i) - \tau \mathrm{Tr}(s) \\ \text{if the eigenvalues } \lambda_1 < \cdots < \lambda_k \text{ of } s \text{ are constant, } \mathcal{E}_i := \mathcal{E}_s(\lambda_i) \\ \text{are holomorphic, and } \varphi \in H^0(\mathrm{Hom}(\mathcal{F}_0, \mathcal{E}_s(0))) \ . \\ \\ \infty \text{ if not .} \end{cases}
$$

Criterion: *A holomorphic pair $(\mathcal{E}, \varphi)$ is semistable with respect to the moment map μ if and only if $\lambda^s(\mathcal{E}) \geqslant 0$ for all $s \in A^0(\mathrm{Herm}(E, h))$.*

Put again $r_i := \mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$ and $m_i = m(\mathcal{E}_i/\mathcal{E}_{i-1})$.
We have the following result:

Theorem 3.5. *For any Hermitian metric h on E, and any non semistable holomorphic pair $(\mathcal{E}, \varphi)$, there exists a unique normalised Hermitian endomorphism $s_{op} \in A^0(\mathrm{Herm}(E, h))$ which satisfies*

$$
\lambda^{s_{op}}(\mathcal{E}) = \inf_{\substack{s \in A^0(\mathrm{Herm}(E,h)) \\ \|s\|=1}} \lambda^s(\mathcal{E}).
$$

It is given by

(i) *If $\mathrm{Im}(\varphi) \subset \mathcal{E}_m$ then*

$$
s_{op} = \frac{1}{\sqrt{\mathrm{Vol}(Y)}\sqrt{\sum_{i=1}^{k} r_i \left[m_i - \tau\right]^2}} \sum_{i=1}^{k} \left[\tau - m_i\right] \mathrm{id}_{F_i} \ .
$$

(ii) *If $\mathrm{Im}(\varphi) \not\subset \mathcal{E}_m$*

$$
s_{op} = \frac{1}{\sqrt{\mathrm{Vol}(Y)}\sqrt{\sum_{\substack{i=1 \\ i\neq m+1}}^{k} r_i \left[m_i - \tau\right]^2}} \sum_{\substack{i=1 \\ i\neq m+1}}^{k} \left[\tau - m_i\right] \mathrm{id}_{F_i} \ ,
$$

where F_i is the h-orthogonal complement of $\mathcal{E}_{i-1}$ in $\mathcal{E}_i$.

Proof. The existence of an optimal destabilizing element s_{op} is proved in the same way as in proposition 2.1: we simply use the fact that inclusion of subsheaves is preserved when going through the limit in the sense of weakly holomorphic subbundles to deal with the additional condition $\mathrm{Im}(\varphi) \subset \mathcal{E}_s(0)$.

Now, let $s \in A^0(\mathrm{Herm}(E, h))$ with constant eigenvalues $\lambda_1 < \cdots < \lambda_k$, such that the filtration given by the $\mathcal{E}_i = \mathcal{E}_s(\lambda_i)$ is holomorphic and $\mathrm{Im}(\varphi) \subset$

$\mathcal{E}_s(0)$. Using the previous notations the expression of the minimal weight map becomes

$$\lambda^s(\mathcal{E}) = \sum_{i=1}^{k} \lambda_i r_i (m_i - \tau).$$

As in the classical situation, we want to minimize this expression with respect to s under the assumption

$$\|s\| = \sqrt{\text{Vol}(Y)}\sqrt{\sum_{i=1}^{k} r_i \lambda_i^2} = 1$$

We use the same idea as in theorem 2.3. Assume first that the filtration by eigenspaces is fixed; we have to minimize the map

$$g_{\{\mathcal{E}_i\}} : (\lambda_1, \cdots, \lambda_k) \mapsto \sum_{i=1}^{k} \lambda_i r_i (m_i - \tau)$$

over the smooth ellipsoid

$$S = \big\{ (\lambda_1, \cdots, \lambda_k) \mid \sum_{i=1}^{k} r_i \lambda_i^2 = 1/\text{Vol}(Y) \big\}.$$

with the two additional conditions

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k. \ (\star 1) \quad \text{and} \quad \text{Im}(\varphi) \subset \mathcal{E}_s(0) \ (\star 2).$$

Once again, resolving the Lagrange problem for $g_{\{\mathcal{E}_i\}}|_S$, we see that there are two critical points of $g_{\{\mathcal{E}_i\}}$ over S which are obtained for

$$\lambda_i = \varepsilon(m_i - \tau), \ 1 \leqslant i \leqslant k \text{ and } \varepsilon = \pm 1 \frac{1}{\sqrt{\text{Vol}(Y)}\sqrt{\sum_{i=1}^{k} r_i (m_i - \tau)^2}}.$$

Let us remark that $g_{\{\mathcal{E}_i\}}$ is negative only for $\varepsilon < 0$.

We have the following lemma:

Lemma 3.6. *Assume s is an optimal destabilizing Hermitian endomorphism and let $\{\mathcal{E}_i\}_{1 \leqslant i \leqslant k}$ its associated filtration. Then there exists l such that*

- (i) $m_1 > \cdots > m_l > \tau > m_{l+2} > \cdots > m_k = m(\mathcal{E})$;
- (ii) $\tau \geq m_{l+1}$ *and the property* $\text{Im}(\varphi) \subset \mathcal{E}_{l+1}$ *holds.*
- (iii) *if moreover* $\text{Im}(\varphi) \subset \mathcal{E}_l$, *then the additional condition* $m_{l+1} > m_{l+2}$ *holds.*

A filtration which satisfies these conditions will be called an admissible filtration.

Proof. These are once again gradient arguments. Assume s is optimal and let l such that $\lambda_1 < \cdots < \lambda_l < 0 \leqslant \lambda_{l+1} < \cdots \lambda_k$.

If $\text{Im}(\varphi) \subset \mathcal{E}_l$, since $\lambda_l < 0$, then the filtration must satisfy $m_1 > \cdots > m_l > \tau \geqslant m_{l+1} > m_{l+2} > \cdots > m_k = m(\mathcal{E})$, otherwise the same gradient argument as in lemma 2.4 contradicts the optimality of s.

If $\text{Im}(\varphi) \not\subset \mathcal{E}_l$, then obviously $\text{Im}(\varphi) \subset \mathcal{E}_{l+1}$ and $\lambda_{l+1} = 0$. Then restricting $g_{\{\mathcal{E}_i\}}$ to $\{\lambda \mid \lambda_{l+1} = 0\}$, a similar argument shows that the associated filtration satisfies i). For the second point, an explicit computation of the gradient of $g_{\{\mathcal{E}_i\}}$, shows that if $m_{l+1} > \tau$, we may move the point λ such

that $g_{\{\mathcal{E}_i\}}$ decreases and $\lambda_{l+1} \leqslant 0$, which contradicts the optimality of s. Thus $\tau \geq m_{l+1}$. ∎

So it is sufficient for our problem to minimize $g_{\{\mathcal{E}_i\}}$ where $\{\mathcal{E}_i\}$ is an admissible filtration. Under this assumption we get:

- If $\mathrm{Im}(\varphi) \subset \mathcal{E}_l$, the minimal value of $g_{\{\mathcal{E}_i\}_{|S}}$ is obtained for

$$(\lambda_1, \cdots, \lambda_k) = \frac{1}{\sqrt{\mathrm{Vol}(Y)}\sqrt{\sum_{i=1}^{k} r_i(m_i - \tau)^2}}(\tau - m_1, \cdots, \tau - m_k)$$

    and

$$g_{\{\mathcal{E}_i\}}(\lambda_1, \cdots, \lambda_k) = -\frac{1}{\sqrt{\mathrm{Vol}(Y)}}\sqrt{\sum_{i=1}^{k} r_i(m_i - \tau)^2}.$$

- If $\mathrm{Im}(\varphi) \not\subset \mathcal{E}_l$, then $g_{\{\mathcal{E}_i\}_{|S}}$ reaches its minimum for

$$(\lambda_1, \cdots, \lambda_k) = \frac{1}{\|\lambda\|}(\tau - m_1, \cdots, \tau - m_l, 0, \tau - m_{l+2}, \cdots, \tau - m_k)$$

    and

$$g_{\{\mathcal{E}_i\}}(\lambda_1, \cdots, \lambda_k) = -\frac{1}{\sqrt{\mathrm{Vol}(Y)}}\sqrt{\sum_{\substack{i=1 \\ i \neq m+1}}^{k} r_i(m_i - \tau)^2}.$$

Now we want to minimize these expressions among all admissible filtrations. As in section 2, we will work on the polygonal line associated to any admissible filtration. We use the same notations as in section 2, and we use the following definition for the energy $E(f)$ of a polygonal line f:

$$E(f) := \sum_{i=1}^{k} r_i(m_i - \tau)^2 = \sum_{i=0}^{l-1} \int_{a_i}^{a_{i+1}} (f'_{\mathcal{D}}(x) - \tau)^2 \mathrm{d}x$$

The difficult point here is that the polygonal line associated to an admissible filtration *may no longer be concave*. We will in fact compare the energy of the different concave parts.

The following technical lemma will play an essential role in our proof:

Lemma 3.7. *Let* $f : [0, r] \to \mathbb{R}$ *and* $g : [0, s] \to \mathbb{R}$ *two distinct concave piecewise affine maps. Assume that*

- (i) $f(0) = g(0)$;
- (ii) $f(x) \geqslant g(x)$ *for all* $x \in [0, min(r, s)]$;
- (iii) $f'(x) \geqslant \tau$ *and* $g'(x) \geqslant \tau$ *where they are defined*;
- (iv) $[f(r) - g(s)]/(r - s) \leqslant \tau$.

Then $E(f) > E(g)$.

Proof. Let us fix $f : [0, r] \to \mathbb{R}$; let $a_0 = 0 < a_1 < \cdots < a_l = s$ be the partition of $[0, s]$ corresponding to $g : [0, s] \to \mathbb{R}$. We will do an induction on the length l of the partition.

For the case $l = 1$, assume first that $s \leqslant r$ then the result is a consequence of lemma 2.8 applied to $f|_{[0,s]}$ and g: using the additional condition 3, the conclusion of the lemma is still correct even if $g(s) \neq f(s)$.

If $s > r$, let us denote by $h : [0, r] \to \mathbb{R}$ the straight line defined by $h(0) = 0$ and $h(r) = f(r)$, then by lemma 2.8, $E(f) \geqslant E(h)$. Denote by h_1 the slope of the line h and g_1 whose of the line g. By hypothesis

$$(h(r) - g(s))/(r - s) \leqslant \tau$$

such that

$$h(r) - r\tau \geqslant g(s) - s\tau$$

and using conditions ii) and iii) we get

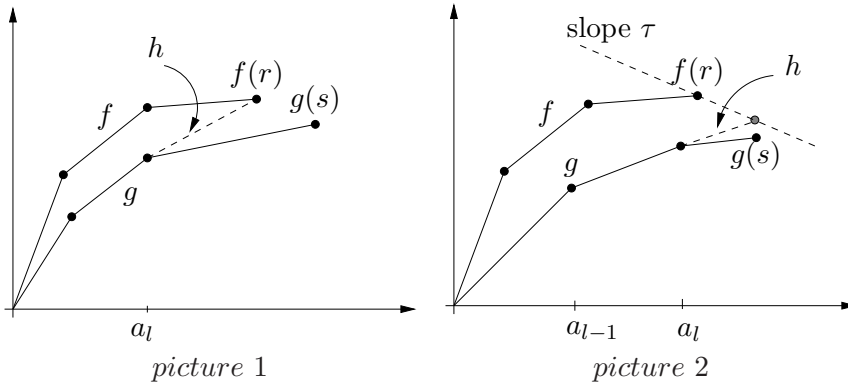$$\frac{1}{r}(h(r) - r\tau)^2 \geqslant \frac{1}{s}(g(s) - s\tau)^2,$$

so that

$$r(h_1 - \tau)^2 \geqslant s(g_1 - \tau)^2.$$

Thus $E(f) \geqslant E(h) \geqslant E(g)$, equality can only occur if $f = g$.

Now assume that the result is proved for a polygonal line of length less or equal than l and let g be a polygonal line of length $l + 1$. Once again, if $s \leqslant r$, we use lemma 2.8 to conclude. Else, we define a new polygonal line h as follows:

- $h_{|[0,a_l]} = g_{|[0,a_l]}$;
- if adding the segment $[g(a_l), f(r)]$ preserves the concavity of h we do so (case 1 below), if not we extend the last segment $[g(a_{l-1}), g(a_l)]$ in order that it reaches the line of slope τ going through $f(r)$ (case 2 below).

$$\textit{picture 1} \qquad\qquad \textit{picture 2}$$

Using the same argument as in the first step of the induction we get easily $E(h) > E(g)$. Then we use lemma 2.8 in the first case and the induction hypothesis applied to f and h in the second case to get $E(f) \geqslant E(h)$. ■

Let $\mathfrak{D} = (0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_m \subset \mathcal{E}_{m+1} \subset \cdots \subset \mathcal{E}_k = \mathcal{E})$ be the generalized Harder-Narasimhan filtration given by theorem 3.1 and $f_{\mathfrak{D}} \in C^0([0, r], \mathbb{R})$ the corresponding piecewise affine map. Put $m_0 = m$ if $\mathrm{Im}(\varphi) \subset \mathcal{E}_m$, $m_0 = m + 1$ otherwise. Then $f_{\mathfrak{D}}$ is not a concave map but admits two concave parts corresponding to $(0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_m)$ and $(\mathcal{E}_{m_0} \subset \cdots \subset \mathcal{E}_k = \mathcal{E})$. Keep in mind that the first m subsheaves of this filtration are just those of

the classical Harder-Narasimhan filtration. Let $D = (0 = \mathcal{E}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_p = \mathcal{E})$ be any admissible filtration. Let us denote by $r'_i := \operatorname{rang}(\mathcal{F}_i/\mathcal{F}_{i-1})$ and $m'_i := m(\mathcal{F}_i/\mathcal{F}_{i-1})$ and assume $m'_j > \tau \geqslant m'_{j+1}$.

Assume that $\operatorname{rang}(\mathcal{F}_j) > \operatorname{rang}(\mathcal{E}_m)$. It follows by the definition of the filtration $\mathfrak{D}$ that $(\deg(\mathcal{E}_m) - \deg(\mathcal{F}_j))/(\operatorname{rang}(\mathcal{E}_m) - \operatorname{rang}(\mathcal{F}_j)) < \tau$: it is an obvious consequence of the fact that the point corresponding to $\mathcal{F}_j$ is located below the classical Harder-Narasimhan polygonal line and the fact that $\mathcal{E}_m$ admits by definition no more type (i) destabilizing subsheaf. Hence applying lemma 3.7 to the first concave part of each filtration, we get

$$\sum_{i=1}^{j} r'_i(\tau - m'_i)^2 \leqslant \sum_{i=1}^{m} r_i(\tau - m_i)^2.$$

If $\operatorname{rang}(\mathcal{F}_j) \leqslant \operatorname{rang}(\mathcal{E}_m)$, then the points corresponding to $\mathcal{F}_i, 1 \leqslant i \leqslant j$ are located below the polygonal line $\mathcal{P}(\mathfrak{D})$ and using again lemma 3.7 we get the same result.

We can apply exactly the same argument to the second part of the filtration. Let $j_0 = j$ if $\operatorname{Im}(\varphi) \subset \mathcal{F}_j$ and $j_0 = j + 1$ in the other case. Then, one can prove that the points corresponding to $(\mathcal{F}_i)_{j_0 \leqslant i \leqslant p}$ satisfy the following conditions :

- for any $i \in \{j_0, \cdots p\}$, if $\operatorname{rang}(\mathcal{F}_i) > \operatorname{rang}(\mathcal{E}_{m_0})$ then $\mathcal{F}_i$ is located below the second part of the generalized Harder-Narasimhan polygonal line (i.e. the points corresponding to $\mathcal{E}_{m_0} \subset \cdots \subset \mathcal{E}$);
- if $\operatorname{rang}(\mathcal{F}_{j_0}) < \operatorname{rang}(\mathcal{E}_{m_0})$, the relation

$$(\deg(\mathcal{F}_{j_0}) - \deg(\mathcal{E}_{m_0}))/(\operatorname{rang}(\mathcal{F}_{j_0}) - \operatorname{rang}(\mathcal{E}_{m_0})) \geqslant \tau$$
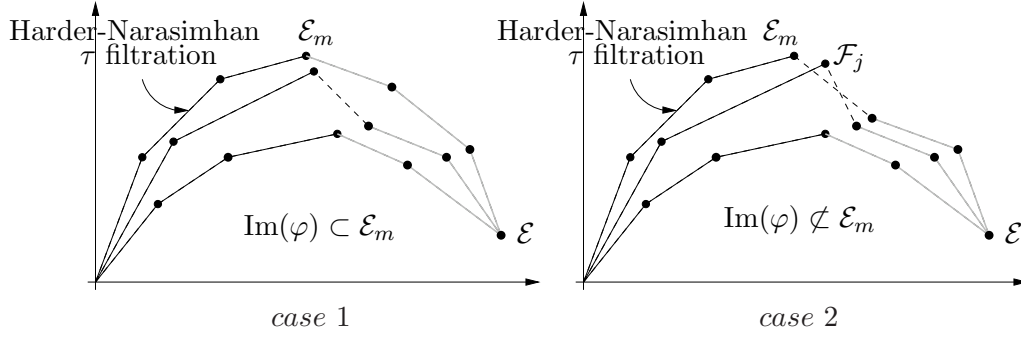
  holds;
- the part of the associated polygonal line corresponding to the indices $\{j_0, \cdots, p\}$ is concave.

The first and the second point can be seen as a Harder-Narasimhan property for subsheaves containing $\operatorname{Im}(\varphi)$. Then using an analogue of lemma 3.7 (in fact a symmetric proposition), we get:

$$\sum_{u=j_0}^{p} r'_u(\tau - m'_u)^2 \leqslant \sum_{u=m_0}^{k} r_u(\tau - m_u)^2.$$

This is exactly that we wanted since in the case where $\operatorname{Im}(\varphi) \not\subset \mathcal{F}_j$, then the expression of $\lambda(\mathcal{F})$ *does not* contain the jth-term (the eigenvalue of the endomorphism must vanish).

In the following picture, the energy of the black and grey parts of the filtrations are respectively compared to the energy of the corresponding parts of the generalized Harder-Narasimhan filtration. The segments drawn with doted lines give no contribution for the energy.

case 1 *case 2*

In either case, we have proved that the minimum of $\lambda(\mathcal{E})$ is achieved for the Hermitian element s_{op} whose associated filtration is the generalized Harder-Narasimhan filtration with the corresponding eigenvalues described in the theorem. ∎

Once again, using the computation of [15], we get:

- if $\mathrm{Im}(\varphi) \subset \mathcal{E}_m$, then it is contained in the elements of the filtration corresponding to (strictly) negative eigenvalues of s_{op}, so that the path $t \mapsto e^{ts_{op}}(\mathcal{E}, \varphi)$ converges as in the classical case to the object

$$(\mathcal{E}_1/\mathcal{E}_0, \cdots, \mathcal{E}_k/\mathcal{E}_{k-1}).$$

- Else, if $\mathrm{Im}(\varphi) \not\subset \mathcal{E}_m$, then it converges to the object

$$(\mathcal{E}_1/\mathcal{E}_0, \cdots, \mathcal{E}_m/\mathcal{E}_{m-1}, (\mathcal{E}_{m+1}/\mathcal{E}_m, \overline{\varphi}), \mathcal{E}_{m+2}/\mathcal{E}_{m+1}, \cdots, \mathcal{E}_k/\mathcal{E}_{k-1}).$$

where $\overline{\varphi}$ is the map induced from φ (remark that in this case, the eigenvalue of s_{op} corresponding to F_m is vanishing).

This illustrates once again our general principle. By theorem 3.1 the limit object is semistable with respect to the gauge group $\prod_{i=1}^{k} Aut(E_i/E_{i-1})$.

This suggests that the principle holds in the whole (infinite dimensional) gauge theoretical framework.

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